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Do electroweak interactions imply six extra time dimensions?

J G Taylor

Department of Mathematics, King's College, University of London, The Strand, London WC2R 2LS, UK

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Abstract. The signature of the metric of extended space–time is investigated in order that spontaneous symmetry breaking is allowed in electroweak interactions embedded in SU(2|1). The only possibility appears to be (+−−−++), in the case of one lepton. Each additional lepton (or pair of flavours) adds an extra pair of time-like dimensions provided each such lepton has mass less than 53 GeV.

There has recently been an upsurge of interest in field theories in higher space–time dimensions, both from the viewpoint of the unification of general relativity and Yang–Mills theories (Cho 1975, Tabensky 1976, Macrae 1978, Mecklenberg 1978) and for extended supergravity (Scherk 1978 and references therein). Even more recently (Ne'eman 1979, Fairlie 1979, Taylor 1979a, Dondi and Jarvis 1979) a similar framework has been used in attempts to embed the electroweak theory of Salam (1968) and Weinberg (1967) in the graded algebra SU(2|1). Since SU(2|1) is the smallest simple graded algebra containing SU(2) × U(1) a prediction of the Weinberg angle θ_w should be possible (Ne'eman 1979, Fairlie 1979) The Higgs scalar fields are described by the extra components of the Yang–Mills gauge field in the higher dimensions, the scalar self-interaction term arising from the Lagrangian quadratic in the Yang–Mills field strengths in the usual fashion (Cho 1975, Tabensky 1976, Macrae 1978, Mecklenberg 1978).

Unfortunately none of the attempts made so far has been completely successful due to various faults. Thus Ne'eman (1979) failed to gauge the theory without violating the spin-statistics theory or include the Higgs scalars satisfactorily, Fairlie (1979) could only incorporate electrons but had the wrong sign for the mass term for the Higgs scalars (the '(mass)^2 disease'), Taylor (1979b) achieved inclusion of more than one lepton but still had the (mass)^2 disease, and Dondi and Jarvis (1979) failed to have the traditional vector meson mass generation mechanism at all, so no lepton masses. No reason was given for the rather *ad hoc* choice of gauge field structure in these references.

The central feature of all these results was that the SU(2) × U(1) electroweak theory embedded in SU(2|1) must have $\theta_w = 30°$, in good agreement with experiment (Salam 1979). It was also possible, on continuation in the mass of the Higgs scalars, to predict (Taylor 1979a) that the mass of the Higgs scalar had to be 150 GeV and that any leptons have to have a maximum mass of 53 GeV. The attractive character of these aspects of the theory indicated that it would be rewarding to construct a satisfactory gauge theory of graded SU(2|1) without the above-mentioned defects.

One step towards this was taken (Taylor 1979b) with a derivation of the original ansatz (Fairlie 1979) for the gauge potential from the requirement of gauge invariance of the graded algebra, together (Pickup and Taylor 1980) with further restriction on the nature of the gauging. However, the $(\text{mass})^2$ disease still persisted. In this paper we show how this disease can be cured completely by choosing the extra dimensions of space–time to be *time-like*, rather than space-like. Thus the space–time extension in this theory is to be regarded as complementary to that of gravity or supergravity (Scherk 1978), where the extension is space-like.

For simplicity we will consider gauging SU(2|1) in six dimensions, with metric $(+---++)$, the extra two dimensions both being time-like. We will denote by Greek letters $\mu, \nu, \ldots$, ordinary space–time indices (with values from 0 to 3) and by Latin indices $m, n, \ldots$ the remainder. The graded algebra SU(2|1) (sometimes denoted a superalgebra) (Corwin *et al* 1975, Freund and Kaplansky 1976) is the set of 3×3 matrices A, with grading on the third row or column, which can be represented as $\left(\begin{smallmatrix} a & ib \\ ib^+ & c \end{smallmatrix}\right)$ where a and c are Hermitian (2×2) and (1×1), respectively, and $\mathrm{Tr}_G A = \mathrm{Tr}\, a - \mathrm{Tr}\, c = 0$. We define $\left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$ to be the even component A_0 of A, $\left(\begin{smallmatrix} 0 & ib \\ ib^+ & 0 \end{smallmatrix}\right)$ to be the odd component A_1 of A. This set of matrices is closed under the bracket operation

$$[A, B] = [A_0, B_0]_- + [A_0, B_1]_- + [A_1, B_0]_- + \mathrm{i}[A_1, B_1]_+. \tag{1}$$

The gauge potential A_M will transform under $u \in \mathrm{SU}(2|1)$ as

$$\delta_u A_M = \partial_M u + \mathrm{i}[A_M, u] \tag{2}$$

and the field strength F_{MN} is defined as

$$F_{MN} = \partial_M A_N - \partial_N A_M + \mathrm{i}[A_M, A_N]. \tag{3}$$

We note that the infinitesimal gauge transformation (2) is taken to be ungraded, so that $\delta_u(AB) = (\delta_u A)B + A(\delta_u B)$ and not $\delta_u(AB) = (\delta_u A)B + (-1)^{ua} A(\delta_u B)$, where $(-1)^{ua} = -1$ only if A and u are both odd and is one otherwise. The gauging (2) differs from that considered in the mathematical literature under the name of a graded derivation (Corwin *et al* 1975, Freund and Kaplansky 1976); it is shown elsewhere (Pickup and Taylor 1979) that graded gauging leads to the same results.

The expected extension of the quadratic Yang–Mills Lagrangian to the graded case would be (Corwin *et al* 1975, Freund and Kaplansky 1976) $\mathrm{Tr}_G(F_{MN}F^{MN})$. However, this is physically unacceptable due to the negative energies associated with the U(1) gauge fields. It is thus (Pickup and Taylor 1980) necessary to attempt to use the ordinary trace to obtain a gauge-invariant Lagrangian. Due to the non-Hermiticity of F_{MN} the most immediate such object would be $\mathrm{Tr}(F_{MN}F^{MN+})$. This was considered earlier (Taylor 1979b) for the metric $(+-----)$ and found satisfactory except for the $(\text{mass})^2$ disease, so requiring analytic continuation in that variable, as mentioned above. In this way the related gauge fields were taken out of SU(2|1) and into Sgl(2|1) so that the theory became a gauge theory of the larger group. This difficulty could have been avoided by using $\mathrm{Tr}(F_{MN}F^{MN})$ except that now the Higgs fields have the wrong sign for their kinetic energy term. Only by choosing the extended space–time metric $(+---++)$ could these terms be made physical.

To see this in detail we analyse the restrictions imposed on A_M in order for $L=\mathrm{Tr}(F_{MN}F^{MN})$ to be invariant under (2). It is straightforward to show that, under (2),

$$\delta_u F_{MN}=\mathrm{i}[F_{MN},u]+\mathrm{i}\{[A_N,\partial_M u]+[\partial_M u,A_N]\}$$
$$+\{[[A_M,A_N],u]-[[A_M,u],A_N]-[A_M,A_N,u]]\}. \tag{4}$$

Invariance of L under (4) requires vanishing of the second term in (4):

$$[A_{N1},\partial_M u_1]_+=0. \tag{5}$$

For $M,N\leqslant 3$ we see that $A_{\mu 1}=0$ for a general space–time-dependent gauge transformation u. We make (5) vanish for $M>3$ by choosing $\partial_m u_1=0$, and so have triviality of the gauge transformation, and so of the gauge potential, in the extra dimensions. In order to remove (5) when $N>3$, $M\leqslant 3$ we must analyse that part of $\delta_u L$ arising from this term in detail. Since (5) is even, we must consider $\mathrm{Tr}([A_{n1},\partial_\mu u_1]_+F^{\mu n 0})$. For this to vanish we require $F_{\mu n 0}=0$, which is true only if A_{n0} is a constant which commutes with all the even generators of SU(2|1) (so commuting with $A_{\mu 0}$). Therefore $A_{n0}\alpha\lambda_{8^1}=(1/\sqrt{3})\,\mathrm{diag}(1,1,2)$.

When u is even the double commutator terms in (4) vanish due to suitable graded Jacobi identities, so that in this case $\delta_u L=\mathrm{Tr}[F_{MN}F^{MN},u_0]_-=0$. The same occurs for odd u and $M,N\leqslant 3$, $M\leqslant 3$, $N>3$ or $M>3$, $N\leqslant 3$ (noting in those latter cases that $F_{MN0}=0$, so only $\partial_{u_1}F_{MN1}$ can contribute to $\delta_{u_1}L$). Thus we consider solely odd u and $M,N>3$ in $\delta_u L$. In this case it can be shown, after some algebraic manipulation, that

$$\begin{aligned}\delta_{u_1}\,&\mathrm{Tr}(F_{mn}F^{mn})\\ &=2\mathrm{Tr}\,u_1\{[[[A_{m1},A_{n1}]_+A_{m1}]_+A_{m0}]_++[[[A_{m1},A_{n1}]_+A_{m1}]_+A_{n0}]_-\\ &\quad+[[([A_{m0},A_{n1}]_-+[A_{m1},A_{n0}]_-),A_{n1}]_+,A_{m1}]_+\\ &\quad+[[([A_{m0},A_{n1}]_-+[A_{m1},A_{n0}]_-),A_{m1}]_+A_{n1}]_+\}.\end{aligned} \tag{6}$$

We see that

$$A_{41}=-A_{51}\qquad A_{40}=A_{50}\qquad\text{or}\qquad A_{41}=A_{51}\qquad A_{40}=-A_{50} \tag{7}$$

gives

$$\delta u_1\,\mathrm{Tr}(F_{44}F^{44}+F_{55}F^{55})=0=\delta u_1\,\mathrm{Tr}(F_{45}F^{45}+F_{54}F^{54}). \tag{8}$$

A more detailed analysis, similar to that in Taylor (1979b), indicates that (7) is very likely the only non-trivial solution of (8), though a rigorous proof of this is still lacking.

Summing our results so far, we can only achieve a gauge-invariant Lagrangian L provided we take (Fairlie 1979, Taylor 1979a, b, Dondi and Jarvis 1979)

$$A_\mu=\frac{g}{2}(A_\mu{}^i\lambda_i+B_\mu\lambda_{8^1}) \tag{9}$$

$$A_m=g\begin{pmatrix} M & \dfrac{\mathrm{i}}{\sqrt{2}}\phi_m\\ \dfrac{\mathrm{i}}{\sqrt{2}}\phi_m^+ & 2M\end{pmatrix} \tag{10}$$

where $\phi_4=-\phi_5$ is an iso-doublet under the SU(2) sub-algebra, $i=1,2,3$ and M is a real constant. The second solution (7) gives identical physical results, so we do not consider it further. We then have

$$F_{\mu\nu} = \tfrac{1}{2}g(F^i_{\mu\nu}(A)\lambda_i + F_{\mu\nu}(B)\lambda_8 1) \tag{11}$$

$$F_m = \frac{\mathrm{i}g}{\sqrt{2}}\begin{pmatrix} 0 & D_\mu\phi_m \\ D_\mu\phi^+_m & 0 \end{pmatrix} = -F_{m\mu} \tag{12}$$

$$F_{mn} = g^2\begin{pmatrix} \tfrac{1}{2}(\phi_m\phi^+_m + \phi_n\phi^+_m) & \dfrac{M}{\sqrt{2}}(\phi_n - \phi_m) \\ \dfrac{M}{\sqrt{2}}(\phi^+_m - \phi^+_n) & \tfrac{1}{2}(\phi^+_n\phi_m + \phi^+_m\phi_n) \end{pmatrix} \tag{13}$$

where $F^i_{\mu\nu}(A)$, $F_{\mu\nu}(B)$ are the usual Yang–Mills fields constructed from $A^i_\mu\lambda_i$ and B_μ, respectively, and D_μ is the gauge-covariant derivative with respect to A_μ in (9). Then, using the metric signature $(+----++)$,

$$-\tfrac{1}{2}g^2L = -\tfrac{1}{4}\left[\sum_i \Gamma^{\mu\nu i}(A)F^i_{\mu\nu}(A) + F_{\mu\nu}(B)F^{\mu\nu}(B)\right] + \sum_m |D_\mu\phi_m|^2$$
$$-4g^2[(\phi^+\phi)^2 - M^2(\phi^+\phi)]. \tag{14}$$

This Lagrangian has correct signs for the Higgs scalar kinetic energy terms, their self-interaction, and their mass term to produce spontaneous symmetry breaking leading to $\langle\phi\rangle_0 \neq 0$. The last of these signs can only be correct if L and not $\mathrm{Tr}(F_{mn}F^+_{mn})$ is considered, and then the first of these signs can only arise by choice of the metric $(+----++)$ and not the previously used signature (Fairlie 1979, Taylor 1979a, b, Dondi and Jarvis 1979) $(+------)$. Since the bose sector of the standard electroweak theory of $SU(2)\times U(1)$ has now been incorporated with $\theta_w = 30°$, this justifies our claim that the extra dimensions must be time-like.

We note in addition the prediction (Taylor 1979a)

$$m_H = 2m_w \sim 152\ \mathrm{GeV}. \tag{15}$$

When we turn to the inclusion of leptons we may follow the use (Taylor 1979a) of two extended time-like dimensions for each new lepton; the lepton Lagrangian must be chosen carefully to preserve gauge invariance. Thus we define the gauge transformation of the lepton spinor ψ with three 8-component spinors to be

$$\delta_u\psi = -\mathrm{i}u\psi. \tag{16}$$

With the gauge-covariant derivative $D_m\psi = \partial_m\psi + \mathrm{i}A_M\psi$, it can be shown readily that

$$\delta(\bar\psi D_M\Gamma^M\psi) = -2\mathrm{i}\bar\psi\gamma_\mu\mu_1\,\partial_\mu\psi + \bar\psi\Gamma_m([A_{m1}, u_1]_- \pm \mathrm{i}[A_{m1}, u_1]_+)\psi \tag{17}$$

where Γ^M are the appropriately extended Dirac matrices; we take $\Gamma_\mu = \gamma_\mu$, $\Gamma_4 = \mathrm{i}\gamma_5\tau_1$, $\Gamma_5 = \mathrm{i}\gamma_5\tau_2$, $\bar\psi = \psi^+\gamma_0$. We see that the first term on the RHS of (17) is only zero provided the grading is defined by chirality, since then u_1 will only have non-zero matrix elements between spinors of different chirality. The second term on the RHS of (17) is then zero for the same reason. Thus $L_1 = \tfrac{1}{2}\mathrm{i}(\bar\psi D_M\Gamma^M\psi)$ is a possible Lagrangian to describe leptons. We note that it comprises the usual expression $\tfrac{1}{2}\mathrm{i}(\bar\psi\gamma^\mu D_\mu\psi)$ and the further term $-\tfrac{1}{2}\bar\psi\Gamma^m A_m\psi$. The first of these terms produces the usual kinetic energy and $SU(2)\times U(1)$ interactions for the lepton if $\psi = (L_1, R_1)$, (L_2R_2), where $(L_i, R_i) = e^{\alpha\gamma^5}(L, R)$, and L is a left-handed doublet under $SU(2)$, R is a right-handed singlet. The remaining term has the value $-\frac{1}{2}i\bar{\psi}\gamma_5(\tau_1 A_4+\tau_2 A_5)\psi$. The purely quadratic term of this can be evaluated with

$$\tau_1=\begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \qquad \tau_2=\begin{pmatrix} 0 & ie^{i\alpha} \\ -ie^{-i\alpha} & 0 \end{pmatrix}$$

to give the value

$$\frac{1}{\sqrt{2}}gM\cos\left(\frac{\pi}{4}-\alpha\right)[\bar{L}\gamma_5 L\cos\alpha_{12}-\bar{L}L\sin\alpha_{12}] \tag{18}$$

where $(L,R)=(\nu_L, l_L, l_R)$ where ν is the neutrino associated with the lepton L. The choice $\alpha_{12}=\alpha_1+\alpha_2=\pi/2$ gives a mass to the lepton of value (Taylor 1979a)

$$m_L=\frac{1}{\sqrt{2}}M_{\mathrm{w}}\cos\left(\frac{\pi}{4}-\alpha\right), \tag{19}$$

this giving the upper bound (Taylor 1979a)

$$m_L\leqslant\frac{1}{\sqrt{2}}M_{\mathrm{w}}. \tag{20}$$

We should remark here that the time-like character of the extra components 4 and 5 is necessary to obtain a real mass for the lepton. In the case that the metric signature had been chosen $(+----)$ it was necessary (Taylor 1979b) to modify the gauging (16) by replacing (U_0+U_1) by (U_0+iU_1) in order to obtain a real mass for the lepton. This was effectively gauging SU(3), so clearly was unsatisfactory in the present graded context. Indeed this feature can be regarded as a further reason for choosing extra components of space–time to be time-like.

We can extend the theory to include further leptons by adding two new extra time dimensions for each new lepton (Taylor 1979a). Thus, for example, in eight dimensions with signature $(+---++++)$ we have $\Gamma_M=R\Gamma_M^{(0)}R^{-1}$, where $\Gamma_\mu^{(0)}=\gamma_\mu$, $\Gamma_4^{(0)}=i\gamma_5\tau_1\rho_3$, $\Gamma_5^{(0)}=i\gamma_5\rho_2\rho_3$, $\Gamma_6^{(0)}=i\gamma_5\rho_1$, $\Gamma_7^{(0)}=i\gamma_5\rho_2$, $R=\mathrm{diag}(e^{i\alpha_1/2}, e^{-i\alpha_1/2}, e^{i\alpha_2/2}, e^{-i\alpha_2/2})$ in the $\tau\otimes\rho$ space. The related gauge potentials can only have two non-zero extended components by extension of (7); if these are taken as A_4 and A_5 then the same structure (9) and (10) results, as does (14). Now the mass term for the leptonic spinor with first eight components electronic, the next eight muonic, each eight components still being chirally graded triplets in SU(2|1), will be the sum of two terms (19) with phases α_e and α_μ respectively. Presently there are three known leptons (α, μ, τ), hence the need for six extra time dimensions.

This theory will fail if the upper bound (20) is violated in the future or if so is the prediction that the mass of the ϕ particle (Higgs particle) is ~150 GeV. Since the Higgs lepton coupling is suppressed by the usual factor (m_l/m_{w}) this prediction may be difficult to test. We note also that since the theory allows only one Higgs doublet there are no axions (Weinberg 1978, Wilczek 1978) present. The inclusion of hadrons in this theory is discussed elsewhere (Taylor 1979c).

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